Formulas

Dot Product

$(x_0, y_0) \cdot (x_1, y_1) = x_0x_1 + \; y_0y_1$
$= |(x_0, y_0)||x_1, y_1|\cos(\theta)$

$$\cos(\theta) = \frac{(x_0, y_0)\cdot(x_1, y_1)}{|(x_0, y_0)||x_1, y_1|} = \frac{a \cdot b}{|a||b|}$$

$a \cdot b > 0 \to \theta$ is acute
$a \cdot b < 0 \to \theta$ is obtuse
$a \cdot b = 0 \to \theta$ is right

$$(a + b) \cdot (c + d) = a \cdot c + b \cdot c + a \cdot d + b \cdot d$$

$$\text{proj}_b\, a = \frac{(a \cdot b)}{|b|^2} b = \left(a \cdot \frac{b}{|b|}\right)\frac{b}{|b|}$$

Lines

- Standard Eq. of Line in $\mathbb{R}^2$ perpendicular to $\vec{n} = (\mathrm{a}, \mathrm{b})$:
$$\vec{n} \cdot (x - x_0, y - y_0) = 0$$

Or equivalently,
$$a(x - x_0) + b(y - y_0) = 0$$

- Vector Eq. of Line in $\mathbb{R}^2$ parallel to $\vec{v} = (a, b)$:
$$(x, y) = (x_0, y_0) + t\vec{v}$$

In $\mathbb{R}^3$ and $\vec{v} = (a, b, c)$ this generalizes to:
$$(x, y, z) = (x_0, y_0, z_0) + t\vec{v}$$

For the *standard parameterization of the line segment*: $0 \le t \le 1$

- The Parametric Eq. for a line can be found from the Vector Eq.
$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tb$$
- The Symmetric Eq. for a line is found by solving for *t*:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes

- Standard Eq. of Plane in $\mathbb{R}^3$ perpendicular to $\vec{n} = (\mathrm{a}, \mathrm{b}, \mathrm{c})$:
$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Or equivalently,
$$a(x - x_0) + b(y - y_0) + c(z - z_0 = 0$$

- Vector Eq. of Plane in $\mathbb{R}^3$ parallel to $\vec{u}$ and $\vec{v}$ (where $\vec{u} \times \vec{v} \neq 0$) :
$$(x, y, z) = (x_0, y_0, z_0) + a\vec{u} + b\vec{v}$$
- The Symmetric Eq. for a line is found by solving for *t*:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Regular

$$\vec{\nabla} f \neq 0$$

Linearly Independent

$$\vec{u} \times \vec{v} \neq 0$$

Cross Product

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$(a, b, c) \times (c, d, e) = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} =$$

$$\vec{\imath}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$|\vec{n} \times \vec{m}| = |\vec{n}||\vec{m}|\sin(\theta)$$

* Direction given by right-hand rule*

$$n \times m = -m \times n$$
$$\vec{p} \times (a\vec{q} + b\vec{r}) = a(\vec{p} \times \vec{q}) + b(\vec{p} \times \vec{r})$$

Partial Derivatives

$$f_{xy} = f_{yx}$$

if f_{xy} and f_{yx} are continuous

$$\vec{\nabla} f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Linear Approximation

$$f(\boldsymbol{x}) \approx f(p) + \vec{\nabla} f(p) \cdot (\boldsymbol{x} - p)$$

Or as a linearization:
$$L_f(\boldsymbol{x}; p) = f(p) + \vec{\nabla} f(p) \cdot (\boldsymbol{x} - p),$$
$$f(\boldsymbol{x}) \approx L_f(\boldsymbol{x}; p)$$

In other words:
$$\Delta f \approx d_p f(\Delta x) = \vec{\nabla} f(p) \cdot (\nabla x)$$

The tangent plane (set) is:
$$z = L_f(\boldsymbol{x}; p)$$

Tangent Plane to Parametric Eq.

$r(u, v) = (x(u, v), y(u, v), z(u, v))$
show r_u and r_v are linearly independent
$(x, y, z) = p + a\overrightarrow{r_u}(u_0, v_0) + b\overrightarrow{r_v}(u_0, v_0)$
Or
$\vec{n} = r_u \times r_v$
$$\vec{n} \cdot \big((x, y, z) - p\big) = 0$$

Hessian Determinant (for checking concavity)

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{|p} = f_{xx}f_{yy} - {f_{xy}}^2$$

$D > 0$, $f_{xx} > 0 \to$ local min.
$D > 0$, $f_{xx} < 0 \to$ local max.
$D < 0 \to$ saddle
$D = 0 \to$ degenerate

Lagrange Multiplier

$\vec{\nabla} f$ is the gradient of the original function.
$\vec{\nabla} g$ is the gradient of the constraint function.
$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

Basic Derivatives

$$\frac{d}{dx}(f + g) = f' + g'$$
$$\frac{d}{dx}(f * g) = f'g + fg'$$
$$\frac{d}{dx}(f/g) = \frac{f'g - g'f}{g^2}$$
$$\frac{d}{dx}(x\hat{\ }a) = ax^{a-1}$$
$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$
$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$
$$\frac{d}{dx}(f(g(x)) = g' * f'(g)$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$$

Chain Rule for Partial Derivatives

$$\frac{\partial f}{\partial t} = \vec{\nabla} f(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{x}}{\partial t}$$

Or
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t} + \cdots$$

Directional Derivative

$\vec{u}$ is a unit vector!
$$D_{\vec{u}} f(p) = d_p f(\vec{u}) = \vec{\nabla} f(p) \cdot \vec{u}$$
$$= |\vec{\nabla} f(p)|\cos(\theta)$$